Lesson 29: The Mathematics Behind a Structured Savings Plan

Classwork

Opening Exercise

Suppose you invested $\$1{,}000$ in an account that paid an annual interest rate of 3% compounded monthly. How much would you have after 1 year?

Example 1

Let $a, ar, ar^2, ar^3, ar^4, \ldots$ be a geometric sequence with first term a and common ratio r. Show that the sum S_n of the first n terms of the geometric series

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \qquad (r \neq 1)$$

is given by the equation

$$S_n = a\left(\frac{1-r^n}{1-r}\right).$$

Exercises 1–3

1. Find the sum of the geometric series $3 + 6 + 12 + 24 + 48 + 96 + 192$.

2. Find the sum of the geometric series $40 + 40(1.005) + 40(1.005)^2 + \cdots + 40(1.005)^{11}$.

3. Describe a situation that might lead to calculating the sum of the geometric series in Exercise 2.

Example 2

A $\$100$ deposit is made at the end of every month for 12 months in an account that earns interest at an annual interest rate of 3% compounded monthly. How much will be in the account immediately after the last payment?

Discussion

An *annuity* is a series of payments made at fixed intervals of time. Examples of annuities include structured savings plans, lease payments, loans, and monthly home mortgage payments. The term annuity sounds like it is only a yearly payment, but annuities often require payments monthly, quarterly, or semiannually. The *future amount of the annuity,* denoted A_f, is the sum of all the individual payments made plus all the interest generated from those payments over the specified period of time.

We can generalize the structured savings plan example above to get a generic formula for calculating the future value of an annuity A_f in terms of the recurring payment R, interest rate i, and number of payment periods n. In the example above, we had a recurring payment of $R = 100$, an interest rate per time period of $i = 0.025$, and 12 payments, so $n = 12$. To make things simpler, we always assume that the payments and the time period in which interest is compounded are at the same time. That is, we do not consider plans where deposits are made halfway through the month with interest compounded at the end of the month.

In the example, the amount A_f of the structured savings plan annuity was the sum of all payments plus the interest accrued for each payment:

$$A_f = R + R(1+i)^1 + R(1+i)^2 + \cdots + R(1+i)^{n-1}.$$

This, of course, is a geometric series with n terms, $a = R$, and $r = 1 + i$, which after substituting into the formula for a geometric series and rearranging is

$$A_f = R\left(\frac{(1+i)^n - 1}{i}\right).$$

Exercises 4–5

4. Write the sum without using summation notation, and find the sum.

   a. $\displaystyle\sum_{k=0}^{5} k$

   b. $\displaystyle\sum_{j=5}^{7} j^2$

   c. $\displaystyle\sum_{i=2}^{4} \frac{1}{i}$

5. Write each sum using summation notation. Do not evaluate the sum.

   a. $1^4 + 2^4 + 3^4 + 4^4 + 5^4 + 6^4 + 7^4 + 8^4 + 9^4$

   b. $1 + \cos(\pi) + \cos(2\pi) + \cos(3\pi) + \cos(4\pi) + \cos(5\pi)$

   c. $2 + 4 + 6 + \cdots + 1000$

Lesson Summary

- **SERIES**: Let $a_1, a_2, a_3, a_4, \ldots$ be a sequence of numbers. A sum of the form

$$a_1 + a_2 + a_3 + \cdots + a_n$$

for some positive integer n is called a *series* (or *finite series*) and is denoted S_n. The a_i's are called the *terms* of the series. The number S_n that the series adds to is called the *sum* of the series.

- **GEOMETRIC SERIES:** A *geometric series* is a series whose terms form a geometric sequence.
- **SUM OF A FINITE GEOMETRIC SERIES**: The sum S_n of the first n terms of the geometric series $S_n = a + ar + \cdots + ar^{n-1}$ (when $r \neq 1$) is given by

$$S_n = a\left(\frac{1 - r^n}{1 - r}\right).$$

The *sum of a finite geometric series* can be written in summation notation as

$$\sum_{k=0}^{n-1} ar^k = a\left(\frac{1 - r^n}{1 - r}\right).$$

- The generic formula for calculating the future value of an annuity A_f in terms of the recurring payment R, interest rate i, and number of periods n is given by

$$A_f = R\left(\frac{(1 + i)^n - 1}{i}\right).$$

Problem Set

1. A car loan is one of the first secured loans most Americans obtain. Research used car prices and specifications in your area to find a reasonable used car that you would like to own (under $10,000). If possible, print out a picture of the car you selected.

   a. What is the year, make, and model of your vehicle?

   b. What is the selling price for your vehicle?

c. The following table gives the monthly cost per $1,000 financed on a 5-year auto loan. Assume you have qualified for a loan with a 5% annual interest rate. What is the monthly cost of financing the vehicle you selected? (A formula is developed to find the monthly payment of a loan in Lesson 30.)

Five-Year (60-month) Loan	
Interest Rate	**Amount per $1,000 Financed**
1.0%	$17.09
1.5%	$17.31
2.0%	$17.53
2.5%	$17.75
3.0%	$17.97
3.5%	$18.19
4.0%	$18.41
4.5%	$18.64
5.0%	$18.87
5.5%	$19.10
6.0%	$19.33
6.5%	$19.56
7.0%	$19.80
7.5%	$20.04
8.0%	$20.28
8.5%	$20.52
9.0%	$20.76

d. What is the gas mileage for your vehicle?

e. Suppose that you drive 120 miles per week and gas costs $4 per gallon. How much does gas cost per month?

2. Write the sum without using summation notation, and find the sum.

a. $\sum_{k=1}^{8} k$

b. $\sum_{k=-8}^{8} k$

c. $\sum_{k=1}^{4} k^3$

d. $\sum_{m=0}^{6} 2m$

e. $\sum_{m=0}^{6} 2m+1$

f. $\sum_{k=2}^{5} \frac{1}{k}$

g. $\sum_{j=0}^{3} (-4)^{j-2}$

h. $\sum_{m=1}^{4} 16\left(\frac{3}{2}\right)^m$

i. $\sum_{j=0}^{3} \frac{105}{2j+1}$

j. $\sum_{p=1}^{3} p \cdot 3^p$

k. $\sum_{j=1}^{6} 100$

l. $\sum_{k=0}^{4} \sin\left(\frac{k\pi}{2}\right)$

m. $\sum_{k=1}^{9} \log\left(\frac{k}{k+1}\right)$

(Hint: You do not need a calculator to find the sum.)

3. Write the sum without using sigma notation. (You do not need to find the sum.)
   a. $\sum_{k=0}^{4} \sqrt{k+3}$
   b. $\sum_{i=0}^{8} x^i$
   c. $\sum_{j=1}^{6} jx^{j-1}$
   d. $\sum_{k=0}^{9} (-1)^k x^k$

4. Write each sum using summation notation.
   a. $1 + 2 + 3 + 4 + \cdots + 1000$
   b. $2 + 4 + 6 + 8 + \cdots + 100$
   c. $1 + 3 + 5 + 7 + \cdots + 99$
   d. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{99}{100}$
   e. $1^2 + 2^2 + 3^2 + 4^2 + \cdots + 10000^2$
   f. $1 + x + x^2 + x^3 + \cdots + x^{200}$
   g. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{49\cdot 50}$
   h. $1\ln(1) + 2\ln(2) + 3\ln(3) + \cdots + 10\ln(10)$

5. Use the geometric series formulas to find the sum of the geometric series.
   a. $1 + 3 + 9 + \cdots + 2187$
   b. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{512}$
   c. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots - \frac{1}{512}$
   d. $0.8 + 0.64 + 0.512 + \cdots + 0.32768$
   e. $1 + \sqrt{3} + 3 + 3\sqrt{3} + \cdots + 243$
   f. $\sum_{k=0}^{5} 2^k$
   g. $\sum_{m=1}^{4} 5\left(\frac{3}{2}\right)^m$
   h. $1 - x + x^2 - x^3 + \cdots + x^{30}$ in terms of x
   i. $\sum_{m=0}^{11} 4^{\frac{m}{3}}$

j. $\sum_{n=0}^{14}\left(\sqrt[5]{6}\right)^n$

k. $\sum_{k=0}^{6} 2\left(\sqrt{3}\right)^k$

6. Let a_i represent the sequence of even natural numbers $\{2,4,6,8,\dots\}$ with $a_1 = 2$. Evaluate the following expressions.

   a. $\displaystyle\sum_{i=1}^{5} a_i$

   b. $\displaystyle\sum_{i=1}^{4} a_{2i}$

   c. $\displaystyle\sum_{i=1}^{5} (a_i - 1)$

7. Let a_i represent the sequence of integers giving the yardage gained per rush in a high school football game $\{3, -2, 17, 4, -8, 19, 2, 3, 3, 4, 0, 1, -7\}$.

   a. Evaluate $\sum_{i=1}^{13} a_i$. What does this sum represent in the context of the situation?

   b. Evaluate $\frac{\sum_{i=1}^{13} a_i}{13}$. What does this expression represent in the context of the situation?

   c. In general, if a_n describes any sequence of numbers, what does $\frac{\sum_{i=1}^{n} a_i}{n}$ represent?

8. Let b_n represent the sequence given by the following recursive formula: $b_1 = 10$, $b_n = 5b_{n-1}$.

   a. Write the first 4 terms of this sequence.

   b. Expand the sum $\sum_{i=1}^{4} b_i$. Is it easier to add this series, or is it easier to use the formula for the sum of a finite geometric sequence? Explain your answer. Evaluate $\sum_{i=1}^{4} b_i$.

   c. Write an explicit form for b_n.

   d. Evaluate $\sum_{i=1}^{10} b_i$.

9. Consider the sequence given by $a_1 = 20$, $a_n = \frac{1}{2} \cdot a_{n-1}$.

   a. Evaluate $\sum_{i=1}^{10} a_i$, $\sum_{i=1}^{100} a_i$, and $\sum_{i=1}^{1000} a_i$.

   b. What value does it appear this series is approaching as n continues to increase? Why might it seem like the series is bounded?

10. The sum of a geometric series with four terms is 60, and the common ratio is $r = \frac{1}{2}$. Find the first term.

11. The sum of the first four terms of a geometric series is 203, and the common ratio is 0.4. Find the first term.

12. The third term in a geometric series is $\frac{27}{2}$, and the sixth term is $\frac{729}{16}$. Find the common ratio.

13. The second term in a geometric series is 10, and the seventh term is $10{,}240$. Find the sum of the first six terms.

14. Find the interest earned and the future value of an annuity with monthly payments of $\$200$ for two years into an account that pays 6% interest per year compounded monthly.

15. Find the interest earned and the future value of an annuity with annual payments of $\$1{,}200$ for 15 years into an account that pays 4% interest per year.

16. Find the interest earned and the future value of an annuity with semiannual payments of $\$1{,}000$ for 20 years into an account that pays 7% interest per year compounded semiannually.

17. Find the interest earned and the future value of an annuity with weekly payments of $\$100$ for three years into an account that pays 5% interest per year compounded weekly.

18. Find the interest earned and the future value of an annuity with quarterly payments of $\$500$ for 12 years into an account that pays 3% interest per year compounded quarterly.

19. How much money should be invested every month with 8% interest per year compounded monthly in order to save up $\$10{,}000$ in 15 months?

20. How much money should be invested every year with 4% interest per year in order to save up $\$40{,}000$ in 18 years?

21. Julian wants to save up to buy a car. He is told that a loan for a car costs $\$274$ a month for five years, but Julian does not need a car presently. He decides to invest in a structured savings plan for the next three years. Every month Julian invests $\$274$ at an annual interest rate of 2% compounded monthly.
    a. How much will Julian have at the end of three years?
    b. What are the benefits of investing in a structured savings plan instead of taking a loan out? What are the drawbacks?

22. An *arithmetic series* is a series whose terms form an arithmetic sequence. For example, $2+4+6+\cdots+100$ is an arithmetic series since $2,4,6,8,\ldots,100$ is an arithmetic sequence with constant difference 2.

    The most famous arithmetic series is $1+2+3+4+\cdots+n$ for some positive integer n. We studied this series in Algebra I and showed that its sum is $S_n=\frac{n(n+1)}{2}$. It can be shown that the general formula for the sum of an arithmetic series $a+(a+d)+(a+2d)+\cdots+[a+(n-1)d]$ is

    $$S_n=\frac{n}{2}[2a+(n-1)d],$$

    where a is the first term and d is the constant difference.

    a. Use the general formula to show that the sum of $1+2+3+\cdots+n$ is $S_n=\frac{n(n+1)}{2}$.

    b. Use the general formula to find the sum of $2+4+6+8+10+\cdots+100$.

23. The sum of the first five terms of an arithmetic series is 25, and the first term is 2. Find the constant difference.

24. The sum of the first nine terms of an arithmetic series is 135, and the first term is 17. Find the ninth term.

25. The sum of the first and 100^{th} terms of an arithmetic series is 101. Find the sum of the first 100 terms.