Lesson 16: Rational and Irrational Numbers

Classwork

Opening Exercise

a. Explain how to use a number line to add the fractions $\frac{7}{5} + \frac{9}{4}$.

b. Convert $\frac{7}{5}$ and $\frac{9}{4}$ to decimals, and explain the process for adding them together.

Exercises

1. According to the calculator, $\log(4) = 0.6020599913\ldots$ and $\log(25) = 1.3979400087\ldots$ Find an approximation of $\log(4) + \log(25)$ to one decimal place, that is, to an accuracy of 10^{-1}.

2. Find the value of $\log(4) + \log(25)$ to an accuracy of 10^{-2}.

3. Find the value of $\log(4) + \log(25)$ to an accuracy of 10^{-8}.

4. Make a conjecture: Is $\log(4) + \log(25)$ a rational or an irrational number?

5. Why is your conjecture in Exercise 4 true?

Remember that the calculator gives the following values: $\log(4) = 0.6020599913\ldots$ and $\log(25) = 1.3979400087\ldots$

6. Find the value of $\log(4) \cdot \log(25)$ to three decimal places.

7. Find the value of $\log(4) \cdot \log(25)$ to five decimal places.

8. Does your conjecture from the above discussion appear to be true?

Lesson Summary

- Irrational numbers occur naturally and frequently.
- The n^{th} roots of most integers and rational numbers are irrational.
- Logarithms of most positive integers or positive rational numbers are irrational.
- We can locate an irrational number on the number line by trapping it between lower and upper estimates. The infinite process of squeezing the irrational number in smaller and smaller intervals locates exactly where the irrational number is on the number line.
- We can perform arithmetic operations such as addition and multiplication with irrational numbers using lower and upper approximations and squeezing the result of the operation in smaller and smaller intervals between two rational approximations to the result.

Problem Set

1. Given that $\sqrt{5} \approx 2.2360679775$ and $\pi \approx 3.1415926535$, find the sum $\sqrt{5} + \pi$ to an accuracy of 10^{-8} without using a calculator.

2. Put the following numbers in order from least to greatest.
$$\sqrt{2}, \pi, 0, e, \frac{22}{7}, \frac{\pi^2}{3}, 3.14, \sqrt{10}$$

3. Find a rational number between the specified two numbers.
   a. $\frac{4}{13}$ and $\frac{5}{13}$
   b. $\frac{3}{8}$ and $\frac{5}{9}$
   c. 1.7299999 and 1.73
   d. $\frac{\sqrt{2}}{7}$ and $\frac{\sqrt{2}}{9}$
   e. π and $\sqrt{10}$

4. Knowing that $\sqrt{2}$ is irrational, find an irrational number between $\frac{1}{2}$ and $\frac{5}{9}$.

5. Give an example of an irrational number between e and π.

6. Given that $\sqrt{2}$ is irrational, which of the following numbers are irrational?
$$\frac{\sqrt{2}}{2}, 2 + \sqrt{2}, \frac{\sqrt{2}}{2\sqrt{2}}, \frac{2}{\sqrt{2}}, \left(\sqrt{2}\right)^2$$

7. Given that π is irrational, which of the following numbers are irrational?

$$\frac{\pi}{2}, \frac{\pi}{2\pi}, \sqrt{\pi}, \pi^2$$

8. Which of the following numbers are irrational?

$$1, 0, \sqrt{5}, \sqrt[3]{64}, e, \pi, \frac{\sqrt{2}}{2}, \frac{\sqrt{8}}{\sqrt{2}}, \cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right)$$

9. Find two irrational numbers x and y so that their average is rational.

10. Suppose that $\frac{2}{3}x$ is an irrational number. Explain how you know that x must be an irrational number. (Hint: What would happen if there were integers a and b so that $x = \frac{a}{b}$?)

11. If r and s are rational numbers, prove that $r + s$ and $r - s$ are also rational numbers.

12. If r is a rational number and x is an irrational number, determine whether the following numbers are always rational, sometimes rational, or never rational. Explain how you know.
    a. $r + x$
    b. $r - x$
    c. rx
    d. x^r

13. If x and y are irrational numbers, determine whether the following numbers are always rational, sometimes rational, or never rational. Explain how you know.
    a. $x + y$
    b. $x - y$
    c. xy
    d. $\frac{x}{y}$