Lesson 13: Changing the Base

Classwork

Exercises

1. Assume that x, a, and b are all positive real numbers, so that $a \neq 1$ and $b \neq 1$. What is $\log_b(x)$ in terms of $\log_a(x)$? The resulting equation allows us to change the base of a logarithm from a to b.

2. Approximate each of the following logarithms to four decimal places. Use the $\boxed{\text{LOG}}$ key on your calculator rather than logarithm tables, first changing the base of the logarithm to 10 if necessary.

   a. $\log(3^2)$

   b. $\log_3(3^2)$

   c. $\log_2(3^2)$

3. In Lesson 12, we justified a number of properties of base-10 logarithms. Working in pairs, justify the following properties of base b logarithms:

   a. $\log_b(1) = 0$

   b. $\log_b(b) = 1$

   c. $\log_b(b^r) = r$

   d. $b^{\log_b(x)} = x$

   e. $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$

f. $\log_b(x^r) = r \cdot \log_b(x)$

g. $\log_b\left(\dfrac{1}{x}\right) = -\log_b(x)$

h. $\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$

4.   Use the $\boxed{\text{LN}}$ and $\boxed{\text{LOG}}$ keys on your calculator to find the value of each logarithm to four decimal places.

   a. $\ln(1)$ $\log(1)$

   b. $\ln(3)$ $\log(3)$

   c. $\ln(10)$ $\log(10)$

   d. $\ln(25)$ $\log(25)$

   e. $\ln(100)$ $\log(100)$

5.   Make a conjecture that compares values of $\log(x)$ to $\ln(x)$ for $x \geq 1$.

6. Justify your conjecture in Exercise 5 using the change of base formula.

7. Write as a single logarithm.

   a. $\ln(4) - 3\ln\left(\frac{1}{3}\right) + \ln(2)$

   b. $\ln(5) + \frac{3}{5}\ln(32) - \ln(4)$

8. Write each expression as a sum or difference of constants and logarithms of simpler terms.

   a. $\ln\left(\frac{\sqrt{5x^3}}{e^2}\right)$

   b. $\ln\left(\frac{(x+y)^2}{x^2+y^2}\right)$

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Lesson Summary

We have established a formula for changing the base of logarithms from b to a:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}.$$

In particular, the formula allows us to change logarithms base b to common or natural logarithms, which are the only two kinds of logarithms that most calculators compute:

$$\log_b(x) = \frac{\log(x)}{\log(b)} = \frac{\ln(x)}{\ln(b)}.$$

We have also established the following properties for base b logarithms. If x, y, a, and b are all positive real numbers with $a \neq 1$ and $b \neq 1$ and r is any real number, then:

1. $\log_b(1) = 0$
2. $\log_b(b) = 1$
3. $\log_b(b^r) = r$
4. $b^{\log_b(x)} = x$
5. $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$
6. $\log_b(x^r) = r \cdot \log_b(x)$
7. $\log_b\left(\frac{1}{x}\right) = -\log_b(x)$
8. $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

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Problem Set

1. Evaluate each of the following logarithmic expressions, approximating to four decimal places if necessary. Use the LN or LOG key on your calculator rather than a table.

    a. $\log_8(16)$
    b. $\log_7(11)$
    c. $\log_3(2) + \log_2(3)$

2. Use logarithmic properties and the fact that $\ln(2) \approx 0.69$ and $\ln(3) \approx 1.10$ to approximate the value of each of the following logarithmic expressions. Do not use a calculator.

    a. $\ln(e^4)$
    b. $\ln(6)$
    c. $\ln(108)$
    d. $\ln\left(\frac{8}{3}\right)$

3. Compare the values of $\log_{\frac{1}{9}}(10)$ and $\log_9\left(\frac{1}{10}\right)$ without using a calculator.

4. Show that for any positive numbers a and b with $a \neq 1$ and $b \neq 1$, $\log_a(b) \cdot \log_b(a) = 1$.

5. Express x in terms of a, e, and y if $\ln(x) - \ln(y) = 2a$.

6. Rewrite each expression in an equivalent form that only contains one base 10 logarithm.
   a. $\log_2(800)$
   b. $\log_x\left(\frac{1}{10}\right)$, for positive real values of $x \neq 1$
   c. $\log_5(12500)$
   d. $\log_3(0.81)$

7. Write each number in terms of natural logarithms, and then use the properties of logarithms to show that it is a rational number.
   a. $\log_9\left(\sqrt{27}\right)$
   b. $\log_8(32)$
   c. $\log_4\left(\frac{1}{8}\right)$

8. Write each expression as an equivalent expression with a single logarithm. Assume x, y, and z are positive real numbers.
   a. $\ln(x) + 2\ln(y) - 3\ln(z)$
   b. $\frac{1}{2}\left(\ln(x + y) - \ln(z)\right)$
   c. $(x + y) + \ln(z)$

9. Rewrite each expression as sums and differences in terms of $\ln(x)$, $\ln(y)$, and $\ln(z)$.
   a. $\ln(xyz^3)$
   b. $\ln\left(\frac{e^3}{xyz}\right)$
   c. $\ln\left(\sqrt{\frac{x}{y}}\right)$

10. Use base-5 logarithms to rewrite each exponential equation as a logarithmic equation, and solve the resulting equation. Use the change of base formula to convert to a base-10 logarithm that can be evaluated on a calculator. Give each answer to 4 decimal places. If an equation has no solution, explain why.

a. $5^{2x} = 20$

b. $75 = 10 \cdot 5^{x-1}$

c. $5^{2+x} - 5^x = 10$

d. $5^{x^2} = 0.25$

11. In Lesson 6, you discovered that $\log(x \cdot 10^k) = k + \log(x)$ by looking at a table of logarithms. Use the properties of logarithms to justify this property for an arbitrary base $b > 0$ with $b \neq 1$. That is, show that $\log_b(x \cdot b^k) = k + \log_b(x)$.

12. Larissa argued that since $\log_2(2) = 1$ and $\log_2(4) = 2$, then it must be true that $\log_2(3) = 1.5$. Is she correct? Explain how you know.

13. Extension: Suppose that there is some positive number b so that

$$\log_b(2) = 0.36$$
$$\log_b(3) = 0.57$$
$$\log_b(5) = 0.84.$$

a. Use the given values of $\log_b(2)$, $\log_b(3)$, and $\log_b(5)$ to evaluate the following logarithms:

  i. $\log_b(6)$

  ii. $\log_b(8)$

  iii. $\log_b(10)$

  iv. $\log_b(600)$

b. Use the change of base formula to convert $\log_b(10)$ to base 10, and solve for b. Give your answer to four decimal places.

14. Use a logarithm with an appropriate base to solve the following exponential equations.

a. $2^{3x} = 16$

b. $2^{x+3} = 4^{3x}$

c. $3^{4x-2} = 27^{x+2}$

d. $4^{2x} = \left(\frac{1}{4}\right)^{3x}$

e. $5^{0.2x+3} = 625$

15. Solve each exponential equation.

    a. $3^{2x} = 81$

    b. $6^{3x} = 36^{x+1}$

    c. $625 = 5^{3x}$

    d. $25^{4-x} = 5^{3x}$

    e. $32^{x-1} = \dfrac{1}{2}$

    f. $\dfrac{4^{2x}}{2^{x-3}} = 1$

    g. $\dfrac{1}{8^{2x-4}} = 1$

    h. $2^x = 81$

    i. $8 = 3^x$

    j. $6^{x+2} = 12$

    k. $10^{x+4} = 27$

    l. $2^{x+1} = 3^{1-x}$

    m. $3^{2x-3} = 2^{x+4}$

    n. $e^{2x} = 5$

    o. $e^{x-1} = 6$

16. In Problem 9(e) of Lesson 12, you solved the equation $3^x = 7^{-3x+2}$ using the logarithm base 10.

    a. Solve $3^x = 7^{-3x+2}$ using the logarithm base 3.

    b. Apply the change of base formula to show that your answer to part (a) agrees with your answer to Problem 9(e) of Lesson 12.

    c. Solve $3^x = 7^{-3x+2}$ using the logarithm base 7.

    d. Apply the change of base formula to show that your answer to part (c) also agrees with your answer to Problem 9(e) of Lesson 12.

17. Pearl solved the equation $2^x = 10$ as follows:

    $$\log(2^x) = \log(10)$$
    $$x \log(2) = 1$$
    $$x = \frac{1}{\log(2)}.$$

    Jess solved the equation $2^x = 10$ as follows:

    $$\log_2(2^x) = \log_2(10)$$
    $$x \log_2(2) = \log_2(10)$$
    $$x = \log_2(10).$$

    Is Pearl correct? Is Jess correct? Explain how you know.