Lesson 12: Properties of Logarithms

Classwork

Opening Exercise

Use the approximation $\log(2) \approx 0.3010$ to approximate the values of each of the following logarithmic expressions.

a. $\log(20)$

b. $\log(0.2)$

c. $\log(2^4)$

Exercises

For Exercises 1–6, explain why each statement below is a property of base-10 logarithms.

1. Property 1: $\log(1) = 0$.

2. Property 2: $\log(10) = 1$.

3. Property 3: For all real numbers r, $\log(10^r) = r$.

4. Property 4: For any $x > 0$, $10^{\log(x)} = x$.

5. Property 5: For any positive real numbers x and y, $\log(x \cdot y) = \log(x) + \log(y)$.
   Hint: Use an exponent rule as well as Property 4.

6. Property 6: For any positive real number x and any real number r, $\log(x^r) = r \cdot \log(x)$.
   Hint: Use an exponent rule as well as Property 4.

7. Apply properties of logarithms to rewrite the following expressions as a single logarithm or number.
   a. $\frac{1}{2}\log(25) + \log(4)$

   b. $\frac{1}{3}\log(8) + \log(16)$

   c. $3\log(5) + \log(0.8)$

8. Apply properties of logarithms to rewrite each expression as a sum of terms involving numbers, $\log(x)$, and $\log(y)$, where x and y are positive real numbers.
   a. $\log(3x^2y^5)$

   b. $\log\left(\sqrt{x^7y^3}\right)$

9. In mathematical terminology, logarithms are *well defined* because if $X = Y$, then $\log(X) = \log(Y)$ for $X, Y > 0$. This means that if you want to solve an equation involving exponents, you can apply a logarithm to both sides of the equation, just as you can take the square root of both sides when solving a quadratic equation. You do need to be careful not to take the logarithm of a negative number or zero.

   Use the property stated above to solve the following equations.

   a. $10^{10x} = 100$

   b. $10^{x-1} = \frac{1}{10^{x+1}}$

   c. $100^{2x} = 10^{3x-1}$

10. Solve the following equations.

   a. $10^{x} = 2^{7}$

   b. $10^{x^2+1} = 15$

   c. $4^{x} = 5^{3}$

Lesson Summary

We have established the following properties for base 10 logarithms, where x and y are positive real numbers and r is any real number:

1. $\log(1) = 0$
2. $\log(10) = 1$
3. $\log(10^r) = r$
4. $10^{\log(x)} = x$
5. $\log(x \cdot y) = \log(x) + \log(y)$
6. $\log(x^r) = r \cdot \log(x)$

Additional properties not yet established are the following:

7. $\log\left(\frac{1}{x}\right) = -\log(x)$
8. $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$

Also, logarithms are well defined, meaning that for $X, Y > 0$, if $X = Y$, then $\log(X) = \log(Y)$.

Problem Set

1. Use the approximate logarithm values below to estimate the value of each of the following logarithms. Indicate which properties you used.

$$\log(2) = 0.3010 \qquad \log(3) = 0.4771$$
$$\log(5) = 0.6990 \qquad \log(7) = 0.8451$$

   a. $\log(6)$
   b. $\log(15)$
   c. $\log(12)$
   d. $\log(10^7)$
   e. $\log\left(\frac{1}{5}\right)$
   f. $\log\left(\frac{3}{7}\right)$
   g. $\log(\sqrt[4]{2})$

2. Let $\log(X) = r$, $\log(Y) = s$, and $\log(Z) = t$. Express each of the following in terms of r, s, and t.
   a. $\log\left(\frac{X}{Y}\right)$
   b. $\log(YZ)$
   c. $\log(X^r)$
   d. $\log(\sqrt[3]{Z})$
   e. $\log\left(\sqrt[4]{\frac{Y}{Z}}\right)$
   f. $\log(XY^2Z^3)$

3. Use the properties of logarithms to rewrite each expression in an equivalent form containing a single logarithm.
   a. $\log\left(\frac{13}{5}\right) + \log\left(\frac{5}{4}\right)$
   b. $\log\left(\frac{5}{6}\right) - \log\left(\frac{2}{3}\right)$
   c. $\frac{1}{2}\log(16) + \log(3) + \log\left(\frac{1}{4}\right)$

4. Use the properties of logarithms to rewrite each expression in an equivalent form containing a single logarithm.
   a. $\log(\sqrt{x}) + \frac{1}{2}\log\left(\frac{1}{x}\right) + 2\log(x)$
   b. $\log(\sqrt[5]{x}) + \log(\sqrt[5]{x^4})$
   c. $\log(x) + 2\log(y) - \frac{1}{2}\log(z)$
   d. $\frac{1}{3}\left(\log(x) - 3\log(y) + \log(z)\right)$
   e. $2(\log(x) - \log(3y)) + 3(\log(z) - 2\log(x))$

5. In each of the following expressions, x, y, and z represent positive real numbers. Use properties of logarithms to rewrite each expression in an equivalent form containing only $\log(x)$, $\log(y)$, $\log(z)$, and numbers.
   a. $\log\left(\frac{3x^2y^4}{\sqrt{z}}\right)$
   b. $\log\left(\frac{42\sqrt[3]{xy^7}}{x^2z}\right)$
   c. $\log\left(\frac{100x^2}{y^3}\right)$
   d. $\log\left(\sqrt{\frac{x^3y^2}{10z}}\right)$
   e. $\log\left(\frac{1}{10x^2z}\right)$

6. Express $\log\left(\frac{1}{x}-\frac{1}{x+1}\right)+\left(\log\left(\frac{1}{x}\right)-\log\left(\frac{1}{x+1}\right)\right)$ as a single logarithm for positive numbers x.

7. Show that $\log\left(x+\sqrt{x^2-1}\right)+\log\left(x-\sqrt{x^2-1}\right)=0$ for $x\geq 1$.

8. If $xy=10^{3.67}$ for some positive real numbers x and y, find the value of $\log(x)+\log(y)$.

9. Solve the following exponential equations by taking the logarithm base 10 of both sides. Leave your answers stated in terms of logarithmic expressions.
   a. $10^{x^2}=320$
   b. $10^{\frac{x}{8}}=300$
   c. $10^{3x}=400$
   d. $5^{2x}=200$
   e. $3^x=7^{-3x+2}$

10. Solve the following exponential equations.
    a. $10^x=3$
    b. $10^y=30$
    c. $10^z=300$
    d. Use the properties of logarithms to justify why x, y, and z form an arithmetic sequence whose constant difference is 1.

11. Without using a calculator, explain why the solution to each equation must be a real number between 1 and 2.
    a. $11^x=12$
    b. $21^x=30$
    c. $100^x=2000$
    d. $\left(\frac{1}{11}\right)^x=0.01$
    e. $\left(\frac{2}{3}\right)^x=\frac{1}{2}$
    f. $99^x=9000$

12. Express the exact solution to each equation as a base-10 logarithm. Use a calculator to approximate the solution to the nearest 1000^{th}.
    a. $11^x = 12$
    b. $21^x = 30$
    c. $100^x = 2000$
    d. $\left(\frac{1}{11}\right)^x = 0.01$
    e. $\left(\frac{2}{3}\right)^x = \frac{1}{2}$
    f. $99^x = 9000$

13. Show that for any real number r, the solution to the equation $10^x = 3 \cdot 10^r$ is $\log(3) + r$.

14. Solve each equation. If there is no solution, explain why.
    a. $3 \cdot 5^x = 21$
    b. $10^{x-3} = 25$
    c. $10^x + 10^{x+1} = 11$
    d. $8 - 2^x = 10$

15. Solve the following equation for n: $A = P(1 + r)^n$.

16. In this exercise, we will establish a formula for the logarithm of a sum. Let $L = \log(x + y)$, where $x, y > 0$.
    a. Show $\log(x) + \log\left(1 + \frac{y}{x}\right) = L$. State as a property of logarithms after showing this is a true statement.
    b. Use part (a) and the fact that $\log(100) = 2$ to rewrite $\log(365)$ as a sum.
    c. Rewrite 365 in scientific notation, and use properties of logarithms to express $\log(365)$ as a sum of an integer and a logarithm of a number between 0 and 10.
    d. What do you notice about your answers to (b) and (c)?
    e. Find two integers that are upper and lower estimates of $\log(365)$.