# Lesson 5: Irrational Exponents—What are $2^{\sqrt{2}}$ and $2^{\pi}$?

## Classwork

### Exercise 1

a. Write the following finite decimals as fractions (you do not need to reduce to lowest terms).

$$1, \quad 1.4, \quad 1.41, \quad 1.414, \quad 1.4142, \quad 1.41421$$

b. Write $2^{1.4}$, $2^{1.41}$, $2^{1.414}$, and $2^{1.4142}$ in radical form ($\sqrt[n]{2^m}$).

c. Use a calculator to compute decimal approximations of the radical expressions you found in part (b) to $5$ decimal places. For each approximation, underline the digits that are also in the previous approximation, starting with $2.00000$ done for you below. What do you notice?

$$2^1 = 2 = 2.00000$$

**Exercise 2**

a. Write six terms of a sequence that a calculator can use to approximate $2^{\pi}$.
(Hint: $\pi = 3.141\,59\ldots$)

b. Compute $2^{3.14}$ and $2^{\pi}$ on your calculator. In which digit do they start to differ?

c. How could you improve the accuracy of your estimate of $2^{\pi}$?

## Problem Set

1. Is it possible for a number to be both rational and irrational?

2. Use properties of exponents to rewrite the following expressions as a number or an exponential expression with only one exponent.

   a. $\left(2^{\sqrt{3}}\right)^{\sqrt{3}}$

   b. $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$

   c. $\left(3^{1+\sqrt{5}}\right)^{1-\sqrt{5}}$

   d. $3^{\frac{1+\sqrt{5}}{2}} \cdot 3^{\frac{1-\sqrt{5}}{2}}$

   e. $3^{\frac{1+\sqrt{5}}{2}} \div 3^{\frac{1-\sqrt{5}}{2}}$

   f. $3^{2\cos^2(x)} \cdot 3^{2\sin^2(x)}$

3. 

   a. Between what two integer powers of 2 does $2^{\sqrt{5}}$ lie?

   b. Between what two integer powers of 3 does $3^{\sqrt{10}}$ lie?

   c. Between what two integer powers of 5 does $5^{\sqrt{3}}$ lie?

4. Use the process outlined in the lesson to approximate the number $2^{\sqrt{5}}$. Use the approximation $\sqrt{5} \approx 2.236\,067\,98$.

   a. Find a sequence of five intervals that contain $\sqrt{5}$ whose endpoints get successively closer to $\sqrt{5}$.

   b. Find a sequence of five intervals that contain $2^{\sqrt{5}}$ whose endpoints get successively closer to $2^{\sqrt{5}}$. Write your intervals in the form $2^r < 2^{\sqrt{5}} < 2^s$ for rational numbers $r$ and $s$.

   c. Use your calculator to find approximations to four decimal places of the endpoints of the intervals in part (b).

   d. Based on your work in part (c), what is your best estimate of the value of $2^{\sqrt{5}}$?

   e. Can we tell if $2^{\sqrt{5}}$ is rational or irrational? Why or why not?

5. Use the process outlined in the lesson to approximate the number $3^{\sqrt{10}}$. Use the approximation $\sqrt{10} \approx 3.162\,277\,7$.

   a. Find a sequence of five intervals that contain $3^{\sqrt{10}}$ whose endpoints get successively closer to $3^{\sqrt{10}}$. Write your intervals in the form $3^r < 3^{\sqrt{10}} < 3^s$ for rational numbers $r$ and $s$.

   b. Use your calculator to find approximations to four decimal places of the endpoints of the intervals in part (a).

   c. Based on your work in part (b), what is your best estimate of the value of $3^{\sqrt{10}}$?

6. Use the process outlined in the lesson to approximate the number $5^{\sqrt{7}}$. Use the approximation $\sqrt{7} \approx 2.645\,751\,31$.
   a. Find a sequence of seven intervals that contain $5^{\sqrt{7}}$ whose endpoints get successively closer to $5^{\sqrt{7}}$. Write your intervals in the form $5^r < 5^{\sqrt{7}} < 5^s$ for rational numbers $r$ and $s$.
   b. Use your calculator to find approximations to four decimal places of the endpoints of the intervals in part (a).
   c. Based on your work in part (b), what is your best estimate of the value of $5^{\sqrt{7}}$?

7. A rational number raised to a rational power can either be rational or irrational. For example, $4^{\frac{1}{2}}$ is rational because $4^{\frac{1}{2}} = 2$, and $2^{\frac{1}{4}}$ is irrational because $2^{\frac{1}{4}} = \sqrt[4]{2}$. In this problem, you will investigate the possibilities for an irrational number raised to an irrational power.
   a. Evaluate $\left(\sqrt{2}\right)^{\left(\sqrt{2}\right)^{\sqrt{2}}}$.
   b. Can the value of an irrational number raised to an irrational power ever be rational?