Lesson 1:　Integer Exponents

Classwork

Opening Exercise

Can you fold a piece of notebook paper in half 10 times?

How thick will the folded paper be?

Will the area of the paper on the top of the folded stack be larger or smaller than a postage stamp?

Exploratory Challenge

　　a.　What are the dimensions of your paper?

　　b.　How thick is one sheet of paper?　Explain how you decided on your answer.

　　c.　Describe how you folded the paper.

d.   Record data in the following table based on the size and thickness of your paper.

Number of Folds	0	1	2	3	4	5	6	7	8	9	10
Thickness of the Stack (in.)											
Area of the Top of the Stack (sq. in.)											

e.   Were you able to fold a piece of notebook paper in half 10 times? Why or why not?

f.   Create a formula that approximates the height of the stack after n folds.

g.   Create a formula that will give you the approximate area of the top after n folds.

h.   Answer the original questions from the Opening Exercise. How do the actual answers compare to your original predictions?

Example 1: Using the Properties of Exponents to Rewrite Expressions

The table below displays the thickness and area of a folded square sheet of gold foil. In 2001, Britney Gallivan, a California high school junior, successfully folded a 100-square-inch sheet of gold foil in half 12 times to earn extra credit in her mathematics class.

Rewrite each of the table entries as a multiple of a power of 2.

Number of Folds	Thickness of the Stack (millionths of a meter)	Thickness Using a Power of 2	Area of the Top (square inches)	Area Using a Power of 2
0	0.28	$0.28 \cdot 2^0$	100	$100 \cdot 2^0$
1	0.56	$0.28 \cdot 2^1$	50	$100 \cdot 2^{-1}$
2	1.12		25	
3	2.24		12.5	
4	4.48		6.25	
5	8.96		3.125	
6	17.92		1.5625	

Example 2: Applying the Properties of Exponents to Rewrite Expressions

Rewrite each expression in the form kx^n, where k is a real number, n is an integer, and x is a nonzero real number.

a. $(5x^5) \cdot (-3x^2)$

b. $\dfrac{3x^5}{(2x)^4}$

c. $\dfrac{3}{(x^2)^{-3}}$

d. $\dfrac{x^{-3}x^4}{x^8}$

Exercises 1–5

Rewrite each expression in the form kx^n, where k is a real number and n is an integer. Assume $x \neq 0$.

1. $2x^5 \cdot x^{10}$

2. $\dfrac{1}{3x^8}$

3. $\dfrac{6x^{-5}}{x^{-3}}$

4. $\left(\dfrac{3}{x^{-22}}\right)^{-3}$

5. $\left(x^2\right)^n \cdot x^3$

Lesson Summary

The Properties of Exponents

For real numbers x and y with $x \neq 0$, $y \neq 0$, and all integers a and b, the following properties hold.

1. $x^a \cdot x^b = x^{a+b}$
2. $(x^a)^b = x^{ab}$
3. $(xy)^a = x^a y^a$
4. $\frac{1}{x^a} = x^{-a}$
5. $\frac{x^a}{x^b} = x^{a-b}$
6. $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$
7. $x^0 = 1$

Problem Set

1. Suppose your class tried to fold an unrolled roll of toilet paper. It was originally 4 in. wide and 30 ft. long. Toilet paper is approximately 0.002 in. thick.

    a. Complete each table, and represent the area and thickness using powers of 2.

Number of Folds n	Thickness After n Folds (in.)
0	
1	
2	
3	
4	
5	
6	

Number of Folds n	Area on Top After n Folds (in^2)
0	
1	
2	
3	
4	
5	
6	

    b. Create an algebraic function that describes the area in square inches after n folds.

    c. Create an algebraic function that describes the thickness in inches after n folds.

2. In the Exit Ticket, we saw the formulas below. The first formula determines the minimum width, W, of a square piece of paper of thickness T needed to fold it in half n times, alternating horizontal and vertical folds. The second formula determines the minimum length, L, of a long rectangular piece of paper of thickness T needed to fold it in half n times, always folding perpendicular to the long side.

$$W = \pi \cdot T \cdot 2^{\frac{3(n-1)}{2}} \qquad L = \frac{\pi T}{6}(2^n + 4)(2^n - 1)$$

Use the appropriate formula to verify why it is possible to fold a 10 inch by 10 inch sheet of gold foil in half 13 times. Use 0.28 millionth of a meter for the thickness of gold foil.

3. Use the formula from Problem 2 to determine if you can fold an unrolled roll of toilet paper in half more than 10 times. Assume that the thickness of a sheet of toilet paper is approximately 0.002 in. and that one roll is 102 ft. long.

4. Apply the properties of exponents to rewrite each expression in the form kx^n, where n is an integer and $x \neq 0$.

    a. $(2x^3)(3x^5)(6x)^2$

    b. $\dfrac{3x^4}{(-6x)^{-2}}$

    c. $\dfrac{x^{-3}x^5}{3x^4}$

    d. $5(x^3)^{-3}(2x)^{-4}$

    e. $\left(\dfrac{x^2}{4x^{-1}}\right)^{-3}$

5. Apply the properties of exponents to verify that each statement is an identity.

    a. $\dfrac{2^{n+1}}{3^n} = 2\left(\dfrac{2}{3}\right)^n$ for integer values of n

    b. $3^{n+1} - 3^n = 2 \cdot 3^n$ for integer values of n

    c. $\dfrac{1}{(3^n)^2} \cdot \dfrac{4^n}{3} = \dfrac{1}{3}\left(\dfrac{2}{3}\right)^{2n}$ for integer values of n

6. Jonah was trying to rewrite expressions using the properties of exponents and properties of algebra for nonzero values of x. In each problem, he made a mistake. Explain where he made a mistake in each part, and provide a correct solution.

    > Jonah's Incorrect Work
    >
    > a. $(3x^2)^{-3} = -9x^{-6}$
    >
    > b. $\dfrac{2}{3x^{-5}} = 6x^5$
    >
    > c. $\dfrac{2x - x^3}{3x} = \dfrac{2}{3} - x^3$

7. If $x = 5a^4$ and $a = 2b^3$, express x in terms of b.

8. If $a = 2b^3$ and $b = -\frac{1}{2}c^{-2}$, express a in terms of c.

9. If $x = 3y^4$ and $y = \frac{s}{2x^3}$, show that $s = 54y^{13}$.

10. Do the following tasks without a calculator.

    a. Express 8^3 as a power of 2.

    b. Divide 4^{15} by 2^{10}.

11. Use powers of 2 to perform each calculation without a calculator or other technology.

    a. $\dfrac{2^7 \cdot 2^5}{16}$

    b. $\dfrac{512000}{320}$

12. Write the first five terms of each of the following recursively defined sequences:

    a. $a_{n+1} = 2a_n$, $a_1 = 3$

    b. $a_{n+1} = (a_n)^2$, $a_1 = 3$

    c. $a_{n+1} = 2(a_n)^2$, $a_1 = x$, where x is a real number Write each term in the form kx^n.

    d. $a_{n+1} = 2(a_n)^{-1}$, $a_1 = y$, $(y \neq 0)$ Write each term in the form kx^n.

13. In Module 1, you established the identity $(1-r)(1 + r + r^2 + \cdots + r^{n-1}) = 1 - r^n$, where r is a real number and n is a positive integer.

    Use this identity to respond to parts (a)–(g) below.

    a. Rewrite the given identity to isolate the sum $1 + r + r^2 + \cdots + r^{n-1}$ for $r \neq 1$.

    b. Find an explicit formula for $1 + 2 + 2^2 + 2^3 + \cdots + 2^{10}$.

    c. Find an explicit formula for $1 + a + a^2 + a^3 + \cdots + a^{10}$ in terms of powers of a.

    d. Jerry simplified the sum $1 + a + a^2 + a^3 + a^4 + a^5$ by writing $1 + a^{15}$. What did he do wrong?

    e. Find an explicit formula for $1 + 2a + (2a)^2 + (2a)^3 + \cdots + (2a)^{12}$ in terms of powers of a.

    f. Find an explicit formula for $3 + 3(2a) + 3(2a)^2 + 3(2a)^3 + \cdots + 3(2a)^{12}$ in terms of powers of a. Hint: Use part (e).

    g. Find an explicit formula for $P + P(1+r) + P(1+r)^2 + P(1+r)^3 + \cdots + P(1+r)^{n-1}$ in terms of powers of $(1+r)$.