Lesson 14: Graphing the Tangent Function

Classwork

Exploratory Challenge/Exercises 1–5

1. Use your calculator to calculate each value of $\tan(x)$ to two decimal places in the table for your group.

Group 1 $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$		**Group 2** $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$		**Group 3** $\left(-\frac{3\pi}{2},-\frac{\pi}{2}\right)$		**Group 4** $\left(\frac{3\pi}{2},\frac{5\pi}{2}\right)$	
x	$\tan(x)$	x	$\tan(x)$	x	$\tan(x)$	x	$\tan(x)$
$-\frac{11\pi}{24}$		$\frac{13\pi}{24}$		$-\frac{35\pi}{24}$		$\frac{37\pi}{24}$	
$-\frac{5\pi}{12}$		$\frac{7\pi}{12}$		$-\frac{17\pi}{12}$		$\frac{19\pi}{12}$	
$-\frac{4\pi}{12}$		$\frac{8\pi}{12}$		$-\frac{16\pi}{12}$		$\frac{20\pi}{12}$	
$-\frac{3\pi}{12}$		$\frac{9\pi}{12}$		$-\frac{15\pi}{12}$		$\frac{21\pi}{12}$	
$-\frac{2\pi}{12}$		$\frac{10\pi}{12}$		$-\frac{14\pi}{12}$		$\frac{22\pi}{12}$	
$-\frac{\pi}{12}$		$\frac{11\pi}{12}$		$-\frac{13\pi}{12}$		$\frac{23\pi}{12}$	
0		π		$-\pi$		2π	
$\frac{\pi}{12}$		$\frac{13\pi}{12}$		$-\frac{11\pi}{12}$		$\frac{25\pi}{12}$	
$\frac{2\pi}{12}$		$\frac{14\pi}{12}$		$-\frac{10\pi}{12}$		$\frac{26\pi}{12}$	
$\frac{3\pi}{12}$		$\frac{15\pi}{12}$		$-\frac{9\pi}{12}$		$\frac{27\pi}{12}$	
$\frac{4\pi}{12}$		$\frac{16\pi}{12}$		$-\frac{8\pi}{12}$		$\frac{28\pi}{12}$	
$\frac{5\pi}{12}$		$\frac{17\pi}{12}$		$-\frac{7\pi}{12}$		$\frac{29\pi}{12}$	
$\frac{11\pi}{24}$		$\frac{35\pi}{24}$		$-\frac{13\pi}{24}$		$\frac{59\pi}{24}$	

Group 5 $\left(-\frac{5\pi}{2}, -\frac{3\pi}{2}\right)$	
x	$\tan(x)$
$-\frac{59\pi}{24}$	
$-\frac{29\pi}{12}$	
$-\frac{28\pi}{12}$	
$-\frac{27\pi}{12}$	
$-\frac{26\pi}{12}$	
$-\frac{25\pi}{12}$	
-2π	
$-\frac{23\pi}{12}$	
$-\frac{22\pi}{12}$	
$-\frac{21\pi}{12}$	
$-\frac{20\pi}{12}$	
$-\frac{19\pi}{12}$	
$-\frac{37\pi}{24}$	

Group 6 $\left(\frac{5\pi}{2}, \frac{7\pi}{2}\right)$	
x	$\tan(x)$
$\frac{61\pi}{24}$	
$\frac{31\pi}{12}$	
$\frac{32\pi}{12}$	
$\frac{33\pi}{12}$	
$\frac{34\pi}{12}$	
$\frac{35\pi}{12}$	
3π	
$\frac{37\pi}{12}$	
$\frac{38\pi}{12}$	
$\frac{39\pi}{12}$	
$\frac{40\pi}{12}$	
$\frac{41\pi}{12}$	
$\frac{83\pi}{24}$	

Group 7 $\left(-\frac{7\pi}{2}, -\frac{5\pi}{2}\right)$	
x	$\tan(x)$
$-\frac{83\pi}{24}$	
$-\frac{41\pi}{12}$	
$-\frac{40\pi}{12}$	
$-\frac{39\pi}{12}$	
$-\frac{38\pi}{12}$	
$-\frac{37\pi}{12}$	
-3π	
$-\frac{35\pi}{12}$	
$-\frac{34\pi}{12}$	
$-\frac{33\pi}{12}$	
$-\frac{32\pi}{12}$	
$-\frac{31\pi}{12}$	
$-\frac{61\pi}{24}$	

Group 8 $\left(\frac{7\pi}{2}, \frac{9\pi}{2}\right)$	
x	$\tan(x)$
$\frac{37\pi}{24}$	
$\frac{43\pi}{12}$	
$\frac{44\pi}{12}$	
$\frac{45\pi}{12}$	
$\frac{46\pi}{12}$	
$\frac{47\pi}{12}$	
4π	
$\frac{49\pi}{12}$	
$\frac{50\pi}{12}$	
$\frac{51\pi}{12}$	
$\frac{52\pi}{12}$	
$\frac{53\pi}{12}$	
$\frac{107\pi}{24}$	

2. The tick marks on the axes provided are spaced in increments of $\frac{\pi}{12}$. Mark the horizontal axis by writing the number of the left endpoint of your interval at the leftmost tick mark, the multiple of π that is in the middle of your interval at the point where the axes cross, and the number at the right endpoint of your interval at the rightmost tick mark. Fill in the remaining values at increments of $\frac{\pi}{12}$.

3. On your plot, sketch the graph of $y = \tan(x)$ on your specified interval by plotting the points in the table and connecting the points with a smooth curve. Draw the graph with a bold marker.

4. What happens to the graph near the edges of your interval? Why does this happen?

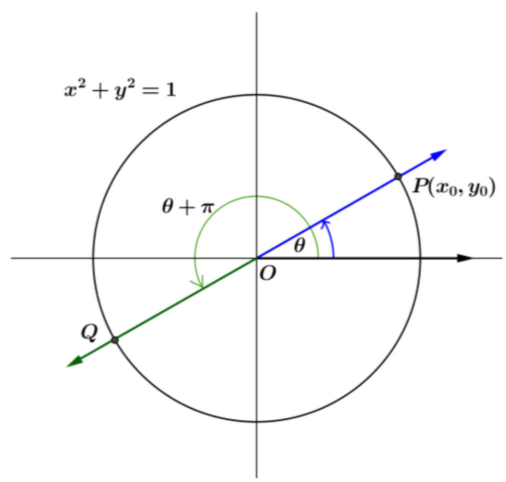
5. When you are finished, affix your graph to the board in the appropriate place, matching endpoints of intervals.

Exploratory Challenge 2/Exercises 6–16

For each exercise below, let $m = \tan(\theta)$ be the slope of the terminal ray in the definition of the tangent function, and let $P = (x_0, y_0)$ be the intersection of the terminal ray with the unit circle after being rotated by θ radians for $0 < \theta < \frac{\pi}{2}$. We know that the tangent of θ is the slope m of $\overleftrightarrow{OP}$.

6. Let Q be the intersection of the terminal ray with the unit circle after being rotated by $\theta + \pi$ radians.

   a. What is the slope of $\overleftrightarrow{OQ}$?



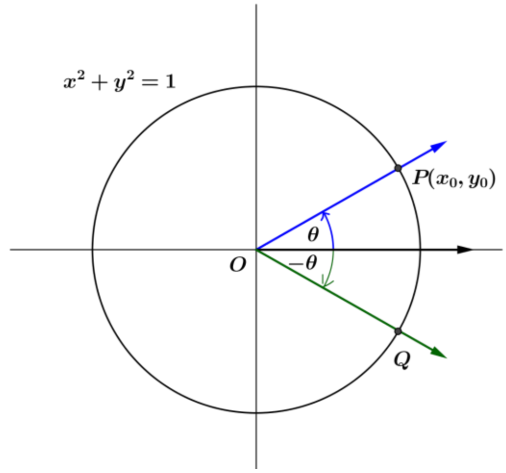
   b. Find an expression for $\tan(\theta + \pi)$ in terms of m.

   c. Find an expression for $\tan(\theta + \pi)$ in terms of $\tan(\theta)$.

   d. How can the expression in part (c) be seen in the graph of the tangent function?

7. Let Q be the intersection of the terminal ray with the unit circle after being rotated by $-\theta$ radians.

   a. What is the slope of $\overleftrightarrow{OQ}$?



   b. Find an expression for $\tan(-\theta)$ in terms of m.

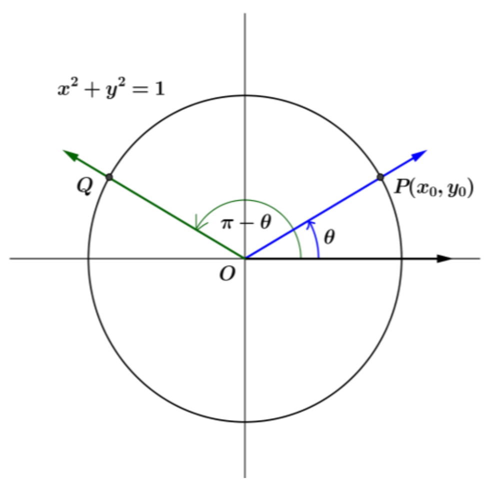
   c. Find an expression for $\tan(-\theta)$ in terms of $\tan(\theta)$.

   d. How can the expression in part (c) be seen in the graph of the tangent function?

8. Is the tangent function an even function, an odd function, or neither? How can you tell your answer is correct from the graph of the tangent function?

9. Let Q be the intersection of the terminal ray with the unit circle after being rotated by $\pi - \theta$ radians.

   a. What is the slope of $\overleftrightarrow{OQ}$?

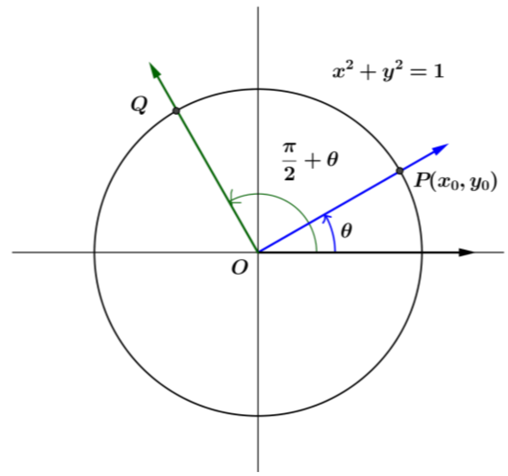


   b. Find an expression for $\tan(\pi - \theta)$ in terms of m.

   c. Find an expression for $\tan(\pi - \theta)$ in terms of $\tan(\theta)$.

10. Let Q be the intersection of the terminal ray with the unit circle after being rotated by $\frac{\pi}{2} + \theta$ radians.

    a. What is the slope of $\overleftrightarrow{OQ}$?

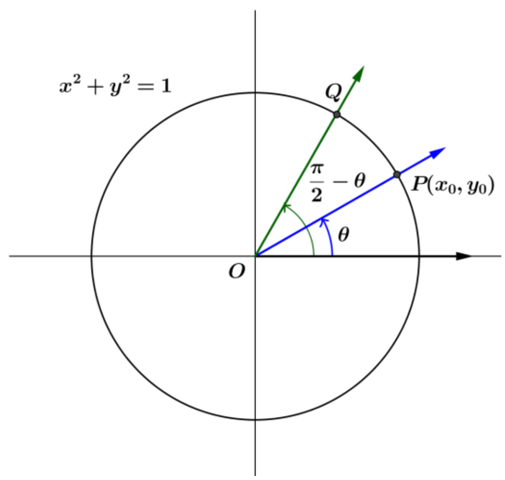


    b. Find an expression for $\tan\left(\frac{\pi}{2} + \theta\right)$ in terms of m.

    c. Find an expression for $\tan\left(\frac{\pi}{2} + \theta\right)$ first in terms of $\tan(\theta)$ and then in terms of $\cot(\theta)$.

11. Let Q be the intersection of the terminal ray with the unit circle after being rotated by $\frac{\pi}{2} - \theta$ radians.

    a. What is the slope of $\overleftrightarrow{OQ}$?



    b. Find an expression for $\tan\left(\frac{\pi}{2} - \theta\right)$ in terms of m.

    c. Find an expression for $\tan\left(\frac{\pi}{2} - \theta\right)$ in terms of $\tan(\theta)$ or other trigonometric functions.

12. Summarize your results from Exercises 6, 7, 9, 10, and 11.

13. We have only demonstrated that the identities in Exercise 12 are valid for $0 < \theta < \frac{\pi}{2}$ because we only used rotations that left point P in the first quadrant. Argue that $\tan\left(-\frac{2\pi}{3}\right) = -\tan\left(\frac{2\pi}{3}\right)$. Then, using similar logic, we could argue that all of the above identities extend to any value of θ for which the tangent (and cotangent for the last two) is defined.

14. For which values of θ are the identities in Exercise 12 valid?

15. Derive an identity for $\tan(2\pi + \theta)$ from the graph.

16. Use the identities you summarized in Exercise 12 to show $\tan(2\pi - \theta) = -\tan(\theta)$ where $\theta \neq \frac{\pi}{2} + k\pi$, for all integers k.

Lesson Summary

The tangent function $\tan(x) = \frac{\sin(x)}{\cos(x)}$ is periodic with period π. The following identities have been established.

- $\tan(x + \pi) = \tan(x)$ for all $x \neq \frac{\pi}{2} + k\pi$, for all integers k.
- $\tan(-x) = -\tan(x)$ for all $x \neq \frac{\pi}{2} + k\pi$, for all integers k.
- $\tan(\pi - x) = -\tan(x)$ for all $x \neq \frac{\pi}{2} + k\pi$, for all integers k.
- $\tan\left(\frac{\pi}{2} + x\right) = -\cot(x)$ for all $x \neq k\pi$, for all integers k.
- $\tan\left(\frac{\pi}{2} - x\right) = \cot(x)$ for all $x \neq k\pi$, for all integers k.
- $\tan(2\pi + x) = \tan(x)$ for all $x \neq \frac{\pi}{2} + k\pi$, for all integers k.
- $\tan(2\pi - x) = -\tan(x)$ for all $x \neq \frac{\pi}{2} + k\pi$, for all integers k.

Problem Set

1. Recall that the cotangent function is defined by $\cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)}$, where $\sin(x) \neq 0$.
   a. What is the domain of the cotangent function? Explain how you know.
   b. What is the period of the cotangent function? Explain how you know.
   c. Use a calculator to complete the table of values of the cotangent function on the interval $(0, \pi)$ to two decimal places.

x	$\cot(x)$
$\frac{\pi}{24}$	
$\frac{\pi}{12}$	
$\frac{2\pi}{12}$	
$\frac{3\pi}{12}$	

x	$\cot(x)$
$\frac{4\pi}{12}$	
$\frac{5\pi}{12}$	
$\frac{\pi}{2}$	

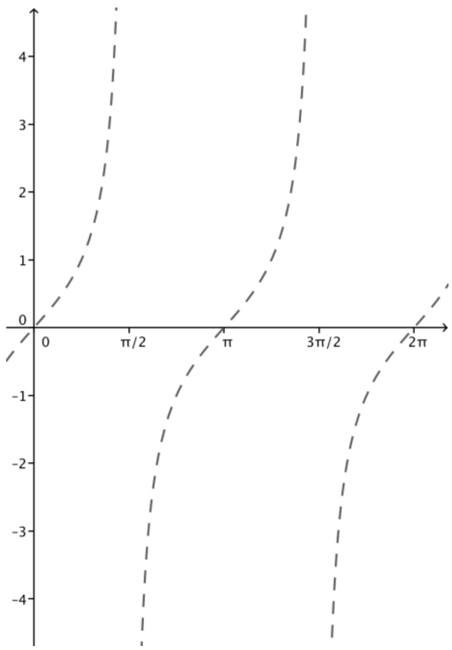
x	$\cot(x)$
$\frac{7\pi}{12}$	
$\frac{8\pi}{12}$	
$\frac{9\pi}{12}$	

x	$\cot(x)$
$\frac{10\pi}{12}$	
$\frac{11\pi}{12}$	
$\frac{23\pi}{24}$	

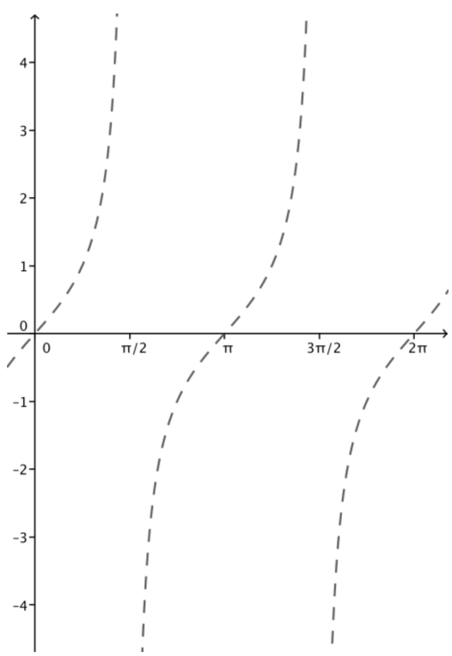
   d. Plot your data from part (c), and sketch a graph of $y = \cot(x)$ on $(0, \pi)$.
   e. Sketch a graph of $y = \cot(x)$ on $(-2\pi, 2\pi)$ without plotting points.
   f. Discuss the similarities and differences between the graphs of the tangent and cotangent functions.
   g. Find all x-values where $\tan(x) = \cot(x)$ on the interval $(0, 2\pi)$.

2. Each set of axes below shows the graph of $f(x) = \tan(x)$. Use what you know about function transformations to sketch a graph of $y = g(x)$ for each function g on the interval $(0, 2\pi)$.

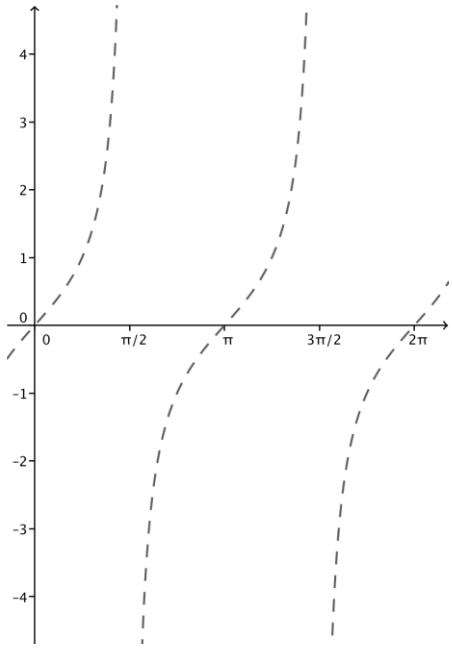
   a. $g(x) = 2\tan(x)$



   b. $g(x) = \frac{1}{3}\tan(x)$



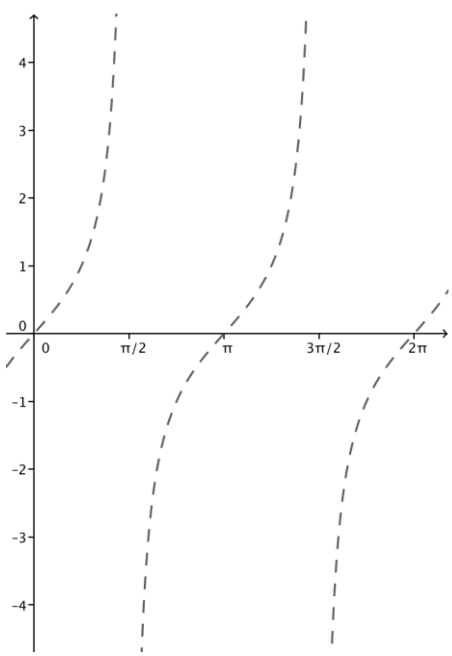
c. $g(x) = -2\tan(x)$



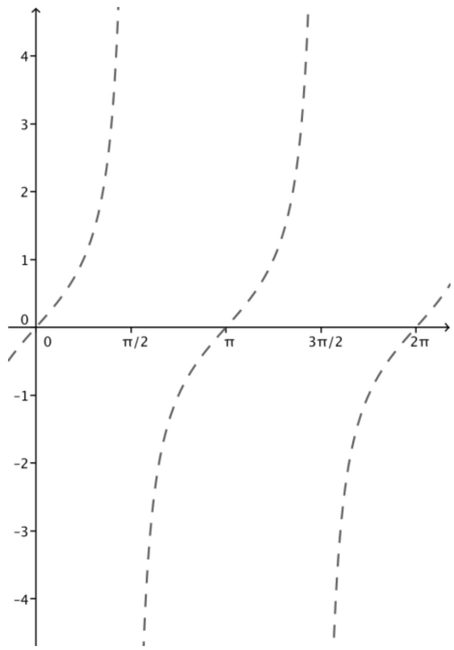
d. How does changing the parameter A affect the graph of $g(x) = A\tan(x)$?

3. Each set of axes below shows the graph of $f(x) = \tan(x)$. Use what you know about function transformations to sketch a graph of $y = g(x)$ for each function g on the interval $(0, 2\pi)$.

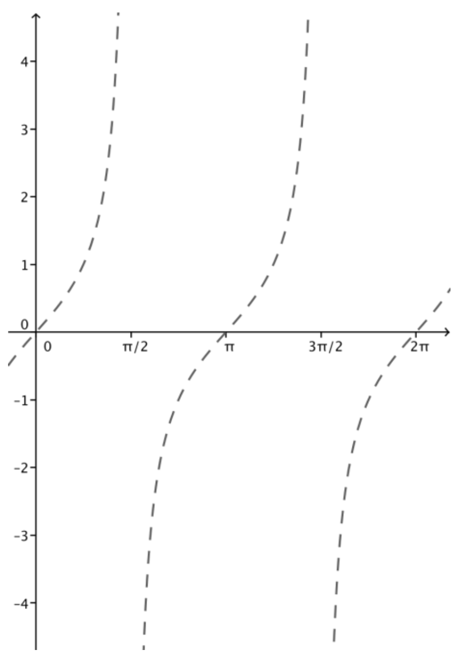
a. $g(x) = \tan\left(x - \frac{\pi}{2}\right)$



b. $g(x) = \tan\left(x - \frac{\pi}{6}\right)$



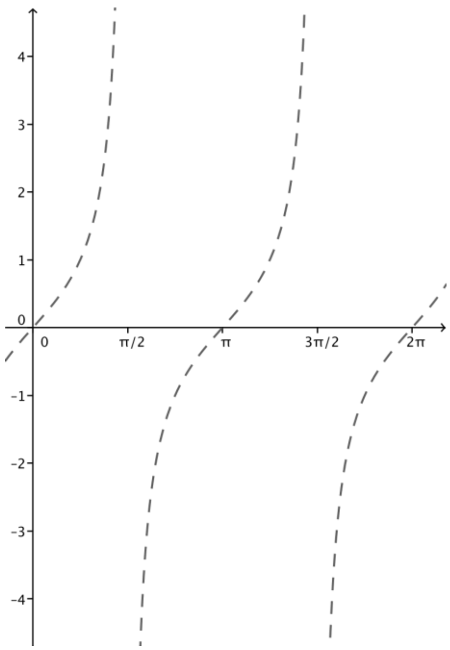
c. $g(x) = \tan\left(x + \frac{\pi}{4}\right)$



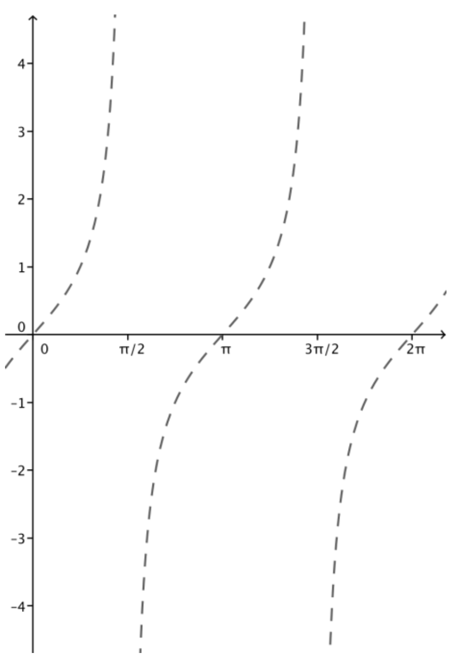
d. How does changing the parameter h affect the graph of $g(x) = \tan(x - h)$?

4. Each set of axes below shows the graph of $f(x) = \tan(x)$. Use what you know about function transformations to sketch a graph of $y = g(x)$ for each function g on the interval $(0, 2\pi)$.

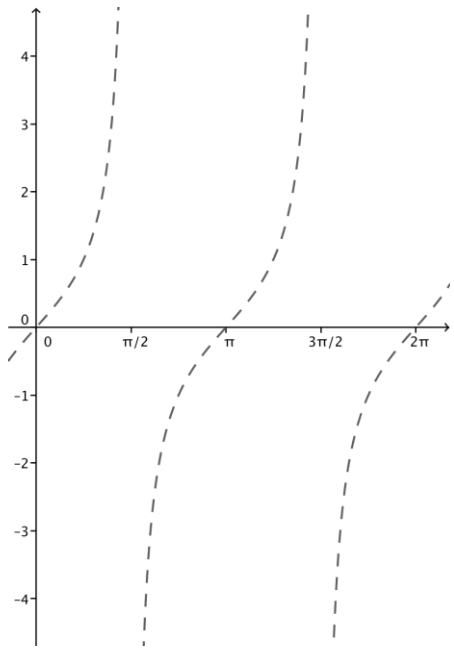
   a. $g(x) = \tan(x) + 1$



   b. $g(x) = \tan(x) + 3$



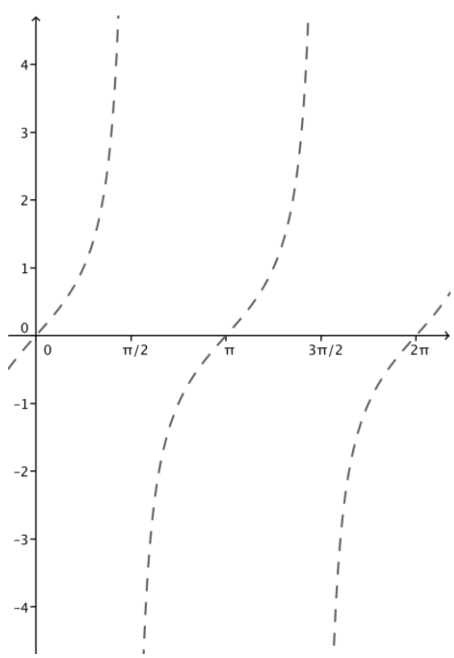
c. $g(x) = \tan(x) - 2$



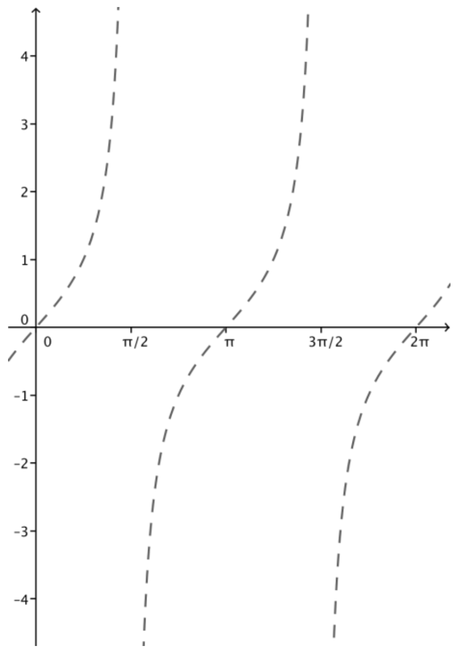
d. How does changing the parameter k affect the graph of $g(x) = \tan(x) + k$?

5. Each set of axes below shows the graph of $f(x) = \tan(x)$. Use what you know about function transformations to sketch a graph of $y = g(x)$ for each function g on the interval $(0, 2\pi)$.

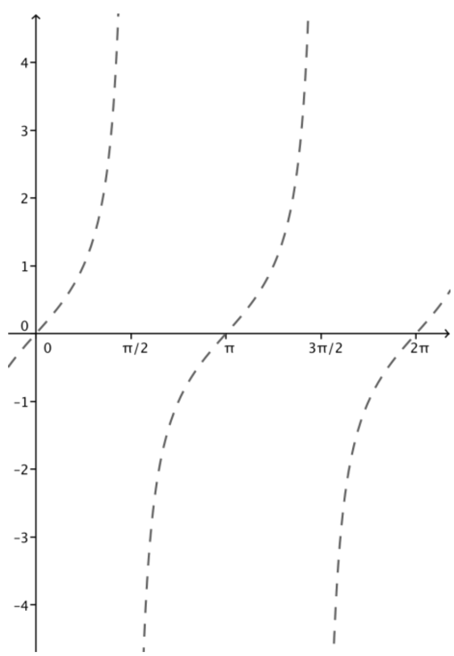
a. $g(x) = \tan(3x)$



b. $g(x) = \tan\left(\frac{x}{2}\right)$



c. $g(x) = \tan(-3x)$



d. How does changing the parameter ω affect the graph of $g(x) = \tan(\omega x)$?

6. Use your knowledge of function transformation and the graph of $y = \tan(x)$ to sketch graphs of the following transformations of the tangent function.
   a. $y = \tan(2x)$
   b. $y = \tan\left(2\left(x - \frac{\pi}{4}\right)\right)$
   c. $y = \tan\left(2\left(x - \frac{\pi}{4}\right)\right) + 1.5$

7. Find parameters A, ω, h, and k so that the graphs of $f(x) = A\tan\big(\omega(x - h)\big) + k$ and $g(x) = \cot(x)$ are the same.