Lesson 5: Extending the Domain of Sine and Cosine to All Real Numbers

Classwork

Opening Exercise

a. Suppose that a group of 360 coworkers pool their money, buying a single lottery ticket every day with the understanding that if any ticket is a winning ticket, the group will split the winnings evenly, and they will donate any leftover money to the local high school. Using this strategy, if the group wins $\$1,000$, how much money will be donated to the school?

b. What if the winning ticket is worth $\$250,000$? Using the same plan as in part (a), how much money will be donated to the school?

c. What if the winning ticket is worth $\$540,000$? Using the same plan as in part (a), how much money will be donated to the school?

Exercises 1–5

1. Find $\cos(405°)$ and $\sin(405°)$. Identify the measure of the reference angle.

2. Find $\cos(840°)$ and $\sin(840°)$. Identify the measure of the reference angle.

3. Find $\cos(1680°)$ and $\sin(1680°)$. Identify the measure of the reference angle.

4. Find $\cos(2115°)$ and $\sin(2115°)$. Identify the measure of the reference angle.

5. Find $\cos(720\,030°)$ and $\sin(720\,030°)$. Identify the measure of the reference angle.

Exercises 6–10

6. Find $\cos(-30°)$ and $\sin(-30°)$. Identify the measure of the reference angle.

7. Find $\cos(-135°)$ and $\sin(-135°)$. Identify the measure of the reference angle.

8. Find $\cos(-1320°)$ and $\sin(-1320°)$. Identify the measure of the reference angle.

9. Find $\cos(-2205°)$ and $\sin(-2205°)$. Identify the measure of the reference angle.

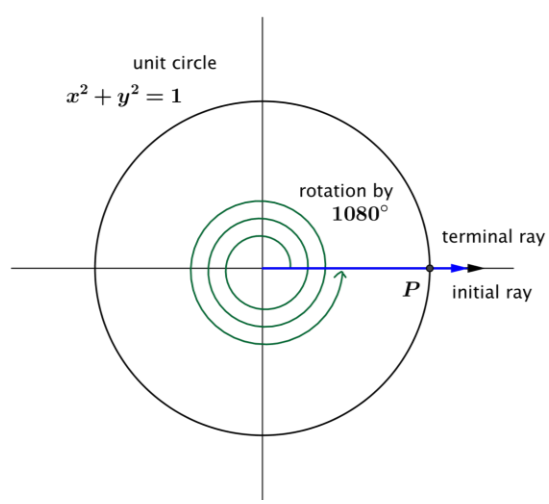
10. Find $\cos(-2835°)$ and $\sin(-2835°)$. Identify the measure of the reference angle.

Discussion

Case 1: What about the values of the sine and cosine function of other amounts of rotation that produce a terminal ray along the positive x-axis, such as $1080°$?

Our definition of a reference angle is the angle formed by the terminal ray and the x-axis, but our terminal ray lies along the x-axis so the terminal ray and the x-axis form a zero angle.

How would we assign values to $\cos(1080°)$ and $\sin(1080°)$?



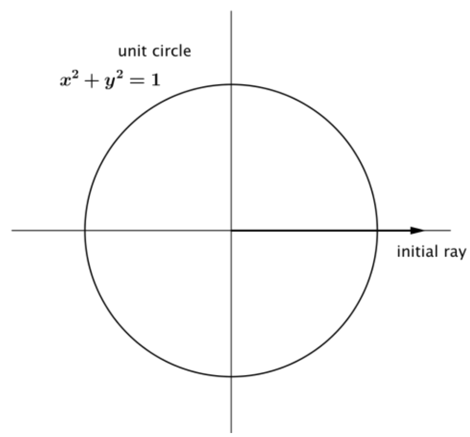
What if we rotated around $24{,}000°$, which is 400 turns? What are $\cos(24000°)$ and $\sin(24000°)$?

State a generalization of these results:

If $\theta = n \cdot 360$, for some integer n, then $\cos(\theta°) =$ ____, and $\sin(\theta°) =$ _____.

Case 2: What about the values of the sine and cosine function of other amounts of rotation that produce a terminal ray along the negative x-axis, such as $540°$?

How would we assign values to $\cos(540°)$ and $\sin(540°)$?

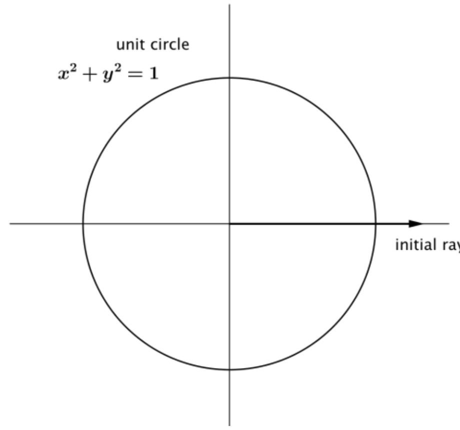


What are the values of $\cos(900°)$ and $\sin(900°)$? How do you know?

State a generalization of these results:

If $\theta = n \cdot 360 + 180$, for some integer n, then $\cos(\theta°) =$ ____, and $\sin(\theta°) =$ _____.

Case 3: What about the values of the sine and cosine function for rotations that are $90°$ more than a number of full turns, such as $-630°$? How would we assign values to $\cos(-630°)$, and $\sin(-630°)$?



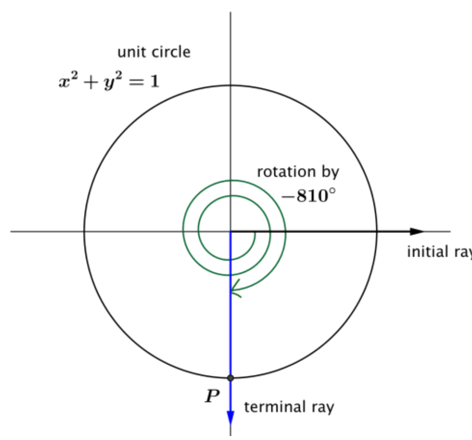
Can we generalize to any rotation that produces a terminal ray along the positive y-axis?

State a generalization of these results:

If $\theta = n \cdot 360 + 90$*, for some integer* n*, then* $\cos(\theta°) =$ ____*, and* $\sin(\theta°) =$ _____.

Case 4: What about the values of the sine and cosine function for rotations whose terminal ray lies along the negative y-axis, such as $-810°$?

How would we assign values to $\cos(-810°)$ and $\sin(-810°)$?



Can we generalize to any rotation that produces a terminal ray along the negative y-axis?

State a generalization of these results:

If $\theta = n \cdot 360 + 270$*, for some integer* n*, then* $\cos(\theta°) =$ ____*, and* $\sin(\theta°) =$ _____.

Discussion

Let θ be any real number. In the Cartesian plane, rotate the initial ray by θ degrees about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ) in the coordinate plane. The value of $\sin(\theta°)$ is y_θ, and the value of $\cos(\theta°)$ is x_θ.

Lesson Summary

In this lesson the definition of the sine and cosine are formalized as functions of a number of degrees of rotation, θ. The initial ray made from the positive x-axis through θ degrees is rotated, going counterclockwise if $\theta > 0$ and clockwise if $\theta < 0$. The point P is defined by the intersection of the terminal ray and the unit circle.

- The value of $\cos(\theta°)$ is the x-coordinate of P.
- The value of $\sin(\theta°)$ is the y-coordinate of P.
- The sine and cosine functions have domain of all real numbers and range $[-1,1]$.

Problem Set

1. Fill in the chart. Write in the measures of the reference angles and the values of the sine and cosine functions for the indicated values of θ.

Number of degrees of rotation, $\boldsymbol{\theta}$	Quadrant	Measure of Reference Angle, in degrees	$\boldsymbol{\cos(\theta°)}$	$\boldsymbol{\sin(\theta°)}$
690				
810				
1560				
1440				
855				
-330				
-4500				
-510				
-135				
-1170				

2. Using geometry, Jennifer correctly calculated that $\sin(15°) = \frac{1}{2}\sqrt{2-\sqrt{3}}$. Based on this information, fill in the chart:

Number of degrees of rotation, $\boldsymbol{\theta}$	**Quadrant**	**Measure of Reference Angle, in degrees**	$\boldsymbol{\cos(\theta°)}$	$\boldsymbol{\sin(\theta°)}$
525				
705				
915				
-15				
-165				
-705				

3. Suppose θ represents a number of degrees of rotation and that $\sin(\theta°) = 0.5$. List the first six possible positive values that θ can take.

4. Suppose θ represents a number of degrees of rotation and that $\sin(\theta°) = -0.5$. List six possible negative values that θ can take.

5. Suppose θ represents a number of degrees of rotation. Is it possible that $\cos(\theta°) = \frac{1}{2}$ and $\sin(\theta°) = \frac{1}{2}$?

6. Jane says that since the reference angle for a rotation through $-765°$ has measure $45°$, then $\cos(-765°) = \cos(45°)$, and $\sin(-765°) = \sin(45°)$. Explain why she is or is not correct.

7. Doug says that since the reference angle for a rotation through $765°$ has measure $45°$, then $\cos(765°) = \cos(45°)$, and $\sin(765°) = \sin(45°)$. Explain why he is or is not correct.