Lesson 23: Comparing Rational Expressions

Classwork

Opening Exercise

Use the slips of paper you have been given to create visual arguments for whether $\frac{1}{3}$ or $\frac{3}{8}$ is larger.

Exercises

We will start by working with positive integers. Let m and n be positive integers. Work through the following exercises with a partner.

1. Fill out the following table.

n	$\frac{1}{n}$
1	
2	
3	
4	
5	
6	

2. Do you expect $\frac{1}{n}$ to be larger or smaller than $\frac{1}{n+1}$? Do you expect $\frac{1}{n}$ to be larger or smaller than $\frac{1}{n+2}$? Explain why.

3. Compare the rational expressions $\frac{1}{n}$, $\frac{1}{n+1}$, and $\frac{1}{n+2}$ for $n = 1$, 2, and 3. Do your results support your conjecture from Exercise 2? Revise your conjecture if necessary.

4. From your work in Exercises 1 and 2, generalize how $\frac{1}{n}$ compares to $\frac{1}{n+m}$, where m and n are positive integers.

5. Will your conjecture change or stay the same if the numerator is 2 instead of 1? Make a conjecture about what happens when the numerator is held constant, but the denominator increases for positive numbers.

Example

x	$\frac{x+1}{x}$	$\frac{x+2}{x+1}$
0.5		
1		
1.5		
2		
5		
10		
100		

Lesson Summary

To compare two rational expressions, find equivalent rational expression with the same denominator. Then we can compare the numerators for values of the variable that do not cause the rational expression to change from positive to negative or vice versa.

We may also use numerical and graphical analysis to help understand the relative sizes of expressions.

Problem Set

1. For parts (a)–(d), rewrite each rational expression as an equivalent rational expression so that all expressions have a common denominator.
   a. $\frac{3}{5}, \frac{9}{10}, \frac{7}{15}, \frac{7}{21}$
   b. $\frac{m}{sd}, \frac{s}{dm}, \frac{d}{ms}$
   c. $\frac{1}{(2-x)^2}, \frac{3}{(2x-5)(x-2)}$
   d. $\frac{3}{x-x^2}, \frac{5}{x}, \frac{2x+2}{2x^2-2}$

2. If x is a positive number, for which values of x is $x < \frac{1}{x}$?

3. Can we determine if $\frac{y}{y-1} > \frac{y+1}{y}$ for all values $y > 1$? Provide evidence to support your answer.

4. For positive x, determine when the following rational expressions have negative denominators.
   a. $\frac{3}{5}$
   b. $\frac{x}{5-2x}$
   c. $\frac{x+3}{x^2+4x+8}$
   d. $\frac{3x^2}{(x-5)(x+3)(2x+3)}$

5. Consider the rational expressions $\frac{x}{x-2}$ and $\frac{x}{x-4}$.
   a. Evaluate each expression for $x = 6$.
   b. Evaluate each expression for $x = 3$.
   c. Can you conclude that $\frac{x}{x-2} < \frac{x}{x-4}$ for all positive values of x? Explain how you know.
   d. Extension: Raphael claims that the calculation below shows that $\frac{x}{x-2} < \frac{x}{x-4}$ for all values of x, where $x \neq 2$ and $x \neq 4$. Where is the error in the calculation?

      Starting with the rational expressions $\frac{x}{x-2}$ and $\frac{x}{x-4}$, we need to first find equivalent rational expressions with a common denominator. The common denominator we will use is $(x-4)(x-2)$. We then have

$$\frac{x}{x-2} = \frac{x(x-4)}{(x-4)(x-2)}$$
$$\frac{x}{x-4} = \frac{x(x-2)}{(x-4)(x-2)}.$$

      Since $x^2 - 4x < x^2 - 2x$ for $x > 0$, we can divide each expression by $(x-4)(x-2)$. We then have $\frac{x(x-4)}{(x-4)(x-2)} < \frac{x(x-2)}{(x-4)(x-2)}$, and we can conclude that $\frac{x}{x-2} < \frac{x}{x-4}$ for all positive values of x.

6. Consider the populations of two cities within the same state where the large city's population is P, and the small city's population is Q. For each of the following pairs, state which of the expressions has a larger value. Explain your reasoning in the context of the populations.
   a. $P + Q$ and P
   b. $\frac{P}{P+Q}$ and $\frac{Q}{P+Q}$
   c. $2Q$ and $P + Q$
   d. $\frac{P}{Q}$ and $\frac{Q}{P}$
   e. $\frac{P}{P+Q}$ and $\frac{1}{2}$
   f. $\frac{P+Q}{P}$ and $P - Q$
   g. $\frac{P+Q}{2}$ and $\frac{P+Q}{Q}$
   h. $\frac{1}{P}$ and $\frac{1}{Q}$